Quintic Matrix Splines

Abstract

One criterion for smoothness of a curve is continuity of its derivatives. Perceptually smooth motion typically has at least two continuous derivatives, often more. Cubic spline curves in general, however, are continuous only in their first derivative. Presented here are a set of spline curves with second derivative continuity, local control, and true interpolation capability.

Introduction

Local control is one desirable quality of a spline. Local control means that each portion of a spline (from one control point to the next control point) can be adjusted independently of all the other portions of the spline. The set of splines that have this property is called "Matrix Splines" due to their being able to be characterized by an equation of the form

$$\underline{T}\,\underline{\underline{M}}\,\underline{K_i} = S_i(t),$$

where $\underline{T}$ is a row vector of descending powers of t, ($[\ldots t^3\, t^2\, t\, 1]$,) $\underline{\underline{M}}$ is the "Magic Matrix" creating the spline, and $\underline{K}$ is a column vector of the control points, often called "knots".

An example of a matrix spline is the uniform cubic B-spline, which has magic matrix

$$\underline{\underline{M}}_{Bspline} = \frac{1}{6}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}.$$

Other desirable qualities of splines include interpolation of control points, derivative continuity, convex hull property, adjustable parameters, and easy computation. We shall choose properties of splines that we want as constraints upon them and derive the appropriate matrix to fulfill those constraints. In particular, we shall require second derivative continuity, which will usually be achieved by setting all second derivatives to zero at the control points.

Derivation

A very useful spline is one that interpolates two positions with zero velocity and acceleration at the endpoints. As a result of these criteria, a particle moving along such a spline slowly exits its first position, gathers speed, then decelerates smoothly to rest at its final position. This movement is called an "ease" in conventional animation; the spline that creates this movement can be called an "ease spline."

We want to specify six constraints: interpolation of control points, zero velocity at the control points, and zero acceleration at the control points. These can be represented

$$\begin{aligned} S(0) &= K_0 \\ S(1) &= K_1 \\ S'(0) &= 0 \\ S'(1) &= 0 \\ S''(0) &= 0 \\ S''(1) &= 0. \end{aligned}$$

Since we have six constraints, we need six free parameters to our spline. Using a polynomial, therefore, we need a quintic. The general quintic and its first and second derivatives are

$$\begin{aligned} S(t) &= at^5 + bt^4 + ct^3 + dt^2 + et + f \\ S'(t) &= 5at^4 + 4bt^3 + 3ct^2 + 2dt + e \\ S''(t) &= 20at^3 + 12bt^2 + 6ct + 2d. \end{aligned}$$

Applying the constraints for the ease spline, we get

$$\begin{aligned} f &= K_0 \\ a + b + c + d + e + f &= K_1 \\ e &= 0 \\ 5a + 4b + 3c + 2d + e &= 0 \\ d &= 0 \\ 20a + 12b + 6c + 2d &= 0. \end{aligned}$$

Solving these simultaneously, we get

$$\begin{aligned} a &= 6(K_1 - K_0) \\ b &= -15(K_1 - K_0) \\ c &= 10(K_1 - K_0) \\ d &= 0 \\ e &= 0 \\ f &= K_0. \end{aligned}$$

This gives us a six by two magic matrix

$$\underline{\underline{M}}_{C^2 Ease} = \begin{bmatrix} -6 & 6 \\ 15 & -15 \\ -10 & 10 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

A use of this spline might be to find a C^2 curve interpolating the unit interval in both x and y dimensions. Substituting into the spline equation yields that curve as

$$S(t) = 6t^5 - 15t^4 + 10t^3.$$

See Figure 1 for a picture of this curve.

Other Results

Another very useful spline is the Hermite Polynomial, also called the "tangent-matching spline." Its control points are the positions to be interpolated and the tangents at those positions. Calling the tangents K_0' and K_1' and placing the control points in the order $[K_0' \; K_0 \; K_1 \; K_1']$, the quintic version of the Hermite Polynomial has magic matrix

$$\underline{\underline{M}}_{C^2 Hermite} = \begin{bmatrix} -6 & -6 & 6 & -3 \\ 15 & 15 & -15 & 7 \\ -10 & -10 & 10 & -4 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

An example of its use might be to find a curve that begins with slope zero and ends the unit interval with slope -1. Substituting into the spline equation, we get

$$S(t) = 9t^5 - 22t^4 + 14t^3.$$

See Figure 2 for a picture of this curve.

Future Extensions

Appendix

The C^1 versions of the Ease spline and Hermite polynomial are

$$\underline{\underline{M}}_{Ease} = \begin{bmatrix} 2 & -2 \\ -3 & 3 \\ 0 & 0 \\ 1 & 0 \end{bmatrix},$$

and

$$\underline{\underline{M}}_{Hermite} = \begin{bmatrix} 1 & 2 & -2 & 1 \\ -2 & -3 & 3 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Figures 3 and 4 compare the C^1 and C^2 versions of these splines graphically, using the same parameters as used above.

References